A New Theoretical Framework for Self-Tuning Control

Weicun Zhang[1], Tianguang Chu[2], and Long Wang[2]

1. Automation Department,
University of Science And Technology Beijing, P.R. China.
weicunzhang @ 263.net

[2] Department of Mechanics and Engineering Science,
Peking University, P.R. China
{ chutg, longwang }@pku.edu.cn

Abstract

This paper presents a new approach---virtual equivalent system method for analysis of self-tuning control (STC) systems, which may lead to a general theory of adaptive control. The method does not rely on specific control law and parameter estimation algorithm, and can thus provide a unified theoretical framework to evaluate the stability and convergence of different STC schemes. Specifically, two theorems are established, based on which some well-known results on stability and convergence of STC systems can be derived in a unified manner.

Keyword: Self-tuning control, stability, convergence, virtual equivalent system.

I. Introduction

Self-tuning adaptive control is a well-known heuristic idea used in the synthesis of identifier or learning-based parameter adaptive control systems. Since the pioneering work of Astrom and Wittenmark in [1], many researchers have devoted their attention to developing stable and convergent self-tuning control systems (see, e.g., [2-27] and the references therein). In particular, Goodwin, Ramadge and Caines [3] proved that the self-tuning control system that comprises stochastic gradient (SG) estimates and minimum-variance control is stable and convergent. Guo and Chen [4] obtained a similar conclusion for the self-tuning control system based on recursive extended least-square (ELS) estimation and minimum-variance control. Also, it has been known that the model reference control, another important ingredient of adaptive control theory, is actually equivalent to self-tuning control (STC).

Despite the fundamental progress achieved so far, a general theory of STC (hence adaptive control) is still absent. As a rule, a well-developed theory should be able to provide deep insights and understanding on the mechanism and properties of a STC system, as well as useful guidelines to the design of new self-tuning schemes for various systems encountered in applications. But, as remarked in [28], “unfortunately, there is no collection of results that can be called a theory of adaptive control in the sense specified.” In the literature much of the existing work has been specific to particular estimation algorithms and control schemes, while only a few attempts have been made towards a unified analysis of the subject from a general perspective (e.g., [13-18]).

In this paper, we intend to introduce a new concept of virtual equivalent system that enables us to develop a unified framework for analysis of the stability and convergence of a STC system, without considering more details about the related estimation algorithms and control schemes. The motivation comes from the observation that for a STC system, there always exists another system whose output is the same of that of the STC system under the same input. We call such a system the virtual equivalent system of a STC system. With the aid of the virtual equivalent system concept, we can convert a usual time-varying STC system into a time-invariant system with a certain time-varying compensation signal, whereby to reduce the complexity and difficulty in analysis of a STC system.

The remainder of the paper is organized as follows. Section 2 introduces the virtual equivalent system, the stability and convergence criteria for deterministic self-tuning control system. Section 3 presents the virtual equivalent system, the stability and convergence criteria for stochastic self-tuning control systems. Section 4 discusses relations between the presented result and the well-known results of [3] and [4]. Finally, some concluding remarks are given in Section 5.

II. Deterministic System

2.1 Construction of Virtual Equivalent System

Consider a deterministic plant ΣP

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k)$$

with

$$A(q^{-1}) = 1 + a_1 q^{-1} + ... + a_n q^{-n}, n \geq 1$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + ... + b_m q^{-m}, m \geq 1$$

The self-tuning control system consists of two parts: a parameter estimator and an adaptive controller. We use $\Sigma P_m(k)$ to denote the estimated model, it can be estimated by any possible algorithms, such as least squares algorithm (LS). Let $\Sigma C(k)$ denote the adaptive controller, it may be designed by any existing principle, such as pole assignment. To be specific, $\Sigma C(k)$ is determined by a certain mapping $f(\cdot)$:

$$\Sigma C(k) = f(\Sigma P_m(k))$$

Accordingly, there exists a special controller ΣC:

$$\Sigma C = f(\Sigma P)$$

which is called a virtual controller because ΣP is unknown.

To get the vector forms of plant ΣP and the estimated model $\Sigma P_m(k)$, we introduce

$$\theta_0^T = [-a_1, ... - a_n, b_0, ..b_m]$$

$$\hat{\theta}^T(k) = [-\hat{a}_1, ... - \hat{a}_n, \hat{b}_0, ..\hat{b}_m]$$

$$\varphi^T(k-d)=[y(k-1),..y(k-n),...,u(k-d),.u(k-d-m)] \tag{2.1}$$

then we have

$$y(k)=\varphi^T(k-d)\theta_0 \tag{2.2}$$

$$y'(k)=\varphi^T(k-d)\hat{\theta}(k)$$

where $y'(k)$ is the output of the estimated model $\Sigma P_m(k)$.

Furthermore, let $\theta_c(k)$ and θ_c be the vector forms of $\Sigma C(k)$ and ΣC respectively.
These two controller's outputs are

$$u(k)=\varphi_c^T(k)\theta_c(k)$$

$$u'(k)=\varphi_c^T(k)\theta_c$$

where

$$\varphi_c{}^T(k)=[y_r(k),...y(k),...,u(k-1),...] \tag{2.3}$$

and the elements of $\varphi_c{}^T(k)$ depend on specific controller.

The self-tuning control system and its virtual equivalent system are shown in Fig.2.1 and Fig.2.2, where $y_r(k)$, $y(k)$ and $u(k)$ are the reference input, output and input of system respectively. In Fig.2.1, the estimator is omitted for simplicity. In Fig.2.2, $\Delta u(k)$ is the output difference between controller $\Sigma C(k)$ and controller ΣC.

$$\Delta u(k)=u(k)-u'(k)$$

$$=\varphi_c^T(k)\theta_c(k)-\varphi_c^T(k)\theta_c \tag{2.4}$$

Notice that $\Delta u(k)$ depends on $\varphi_c(k)$, that is to say it is dependent on the input and output of the original STC system (Fig.2.1). It cannot be calculated from equation (2.4), because θ_c is unknown. But it can be deduced in another way. Actually, $\Delta u(k)$ is a function of $\Delta y(k)$ (see Discussion, for example).

From Fig.2.2, we see that $y(k)$ comprises two parts. One part is the output with the reference input $y_r(k)$ exerted on the system; another part is the output with $\Delta u(k)$ exerted on the system.

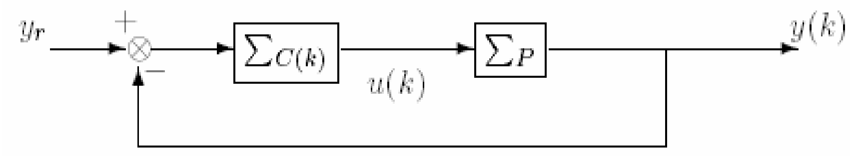

Fig.2.1. Self-tuning Control (Deterministic)

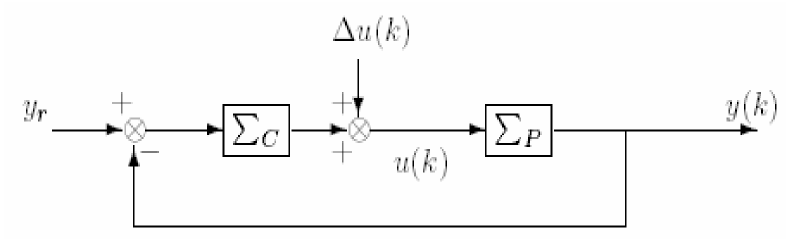

Fig. 2.2. Virtual equivalent system (Deterministic).

2.2 Main Results

The main results are stated as below:

Theorem 2.1:

If a self-tuning control system has the following properties:

1) The controller is well defined such that $(\Sigma C, \Sigma P)$ constitutes a stable closed-loop system.

2) The estimator satisfies

$$y(k) - y'(k) = \varphi^T(k-d)\hat{\theta}(k) - \varphi^T(k-d)\theta_0 = o(1 + \|\varphi(k-d)\|)$$

3) The mapping from estimated parameters into controller parameters satisfies

$$\varphi_c^T(k)\hat{\theta}_c(k) - \varphi_c^T(k)\theta_c = O(\|\varphi^T(k-d)\hat{\theta}(k) - \varphi^T(k-d)\theta_0\|)$$

Then the self-tuning control system is stable and convergent.

Corollary:

If a self-tuning control system has the following properties:

1) The controller is well defined such that $(\Sigma C, \Sigma P)$ constitutes a stable closed-loop system.

2) The estimator satisfies

$$y(k) - y'(k) = \varphi^T(k-d)\hat{\theta}(k) - \varphi^T(k-d)\theta_0 = o(1)$$

3) The mapping from estimated parameters into controller parameters satisfies

$$\varphi_c^T(k)\hat{\theta}_c(k) - \varphi_c^T(k)\theta_c = O(\|\varphi^T(k-d)\hat{\theta}(k) - \varphi^T(k-d)\theta_0\|)$$

Then the self-tuning control system is stable and convergent.

2.3 Proof of Theorem

From conditions 2) and 3), we have

$$\Delta u(k) = o(1 + \|\varphi(k - d)\|) \tag{2.5}$$

Furthermore, we decompose virtual equivalent system Fig.2.2 into two subsystems as shown in Fig.2.3 and Fig.2.4.

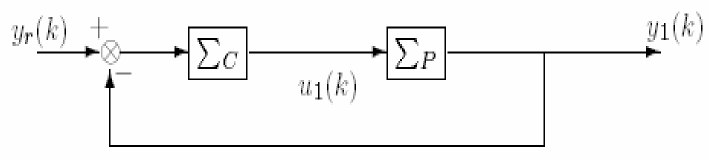

Fig. 2.3. Subsystem 1

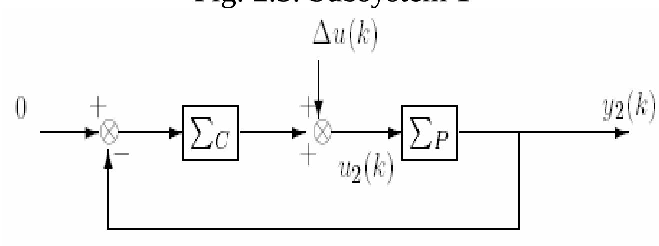

Fig. 2.4. Subsystem 2

By superposition property, we have

$$y(k) = y_1(k) + y_2(k) \tag{2.6}$$

$$u(k) = u_1(k) + u_2(k) \tag{2.7}$$

First we treat subsystem 1 (Fig.2.3) to observe $y_1(k)$ and $u_1(k)$, obviously it is a time-invariant deterministic system, and condition 1) guarantees the closed-loop system is stable and convergent.

We get

$$\|u_1(k)\| < \infty \tag{2.8}$$

$$\|y_1(k)\| < \infty \tag{2.9}$$

and

$$y_1(k) \to y_r(k)$$

Second we treat subsystem 2 (Fig.2.4) to observe $y_2(k)$ and $u_2(k)$. For it is a stable closed-loop system and its input is given by (2.5).

So we have

$$y_2(k) = o(1 + \|\varphi(k-d)\|)$$

$$u_2(k) = o(1 + \|\varphi(k-d)\|)$$

Considering (2.6) and (2.7)

$$y(k) = y_1(k) + o(1 + \|\varphi(k-d)\|) \tag{2.10}$$

$$u(k) = u_1(k) + o(1 + \|\varphi(k-d)\|) \tag{2.11}$$

From these two equations we can conclude that virtual equivalent system (shown in Fig.2.2) is stable, i.e. $\|\varphi(k-d)\| < \infty$. We use counterevidence to prove it. Suppose sequence $\{\|\varphi(k-d)\|\}$ is not bounded, then there is a subsequence $\{\|\varphi(l-d)\|\}$ such that

$$\|\varphi(l-d)\| \to \infty$$

Then, from equations (2.8), (2.9), (2.10) and (2.11), we have

$$\frac{y(l-1)}{1+\|\varphi(l-d)\|} \to 0$$

......

$$\frac{y(l-n)}{1+\|\varphi(l-d)\|} \to 0$$

$$\frac{u(l-d)}{1+\|\varphi(l-d)\|} \to 0$$

......

$$\frac{u(l-d-m)}{1+\|\varphi(l-d)\|} \to 0$$

Make sums on both sides of the above equations, we obtain

$$\frac{\|\varphi(l-d)\|}{1+\|\varphi(l-d)\|} \to 0$$

Obviously, it is contradictory to $\|\varphi(l-d)\| \to \infty$. Then we conclude

$$\|\varphi(k-d)\| < \infty$$

Combining this with equations (2.10) and (2.11), we get

$$y(k) \to y_1(k)$$

$$u(k) \to u_1(k)$$

It follows that

$$y(k) \to y_r(k)$$

$$\|u(k)\| < \infty$$

$$\|y(k)\| < \infty$$

This completes the proof of theorem 2.1.

III. Stochastic System

3.1 Construction of Virtual Equivalent System

Consider a stochastic plant ΣP :

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k) + C(q^{-1})\omega(k) \tag{3.1}$$

with

$$A(q^{-1}) = 1 + a_1q^{-1} + ... + a_nq^{-n}, n \geq 1$$

$$B(q^{-1}) = b_0 + b_1q^{-1} + ... + b_mq^{-m}, m \geq 1$$

$$C(q^{-1}) = c_0 + c_1q^{-1} + ... + c_rq^{-r}, r \geq 1 \tag{3.2}$$

The STC system for ΣP consists of two parts: parameter estimator and control law. The block diagram of the STC system is shown in Fig.3.1, where $y_r(k)$, $y(k)$, $u(k)$ and $\omega(k)$ are the reference input, output, input and noise of system respectively. For simplicity, the estimator is omitted. We denote the estimated model by $\Sigma P_m(k)$, it can be obtained by any available estimate algorithms, such as stochastic gradient (SG), least squares (LS) or extended least squares (ELS). The controller obtained from $\Sigma P_m(k)$, according to some control law, is denoted by $\Sigma C(k)$. The virtual controller ΣC is determined by ΣP according to the same control law as from $\Sigma P_m(k)$ to $\Sigma C(k)$.

We introduce the following notations:

$$\theta_0{}^T = [-a_1, \cdots - a_n, b_0, \cdots b_m, c_0, \cdots c_r] \tag{3.3}$$

$$\hat{\theta}^T(k) = [-\hat{a}_1, \cdots - \hat{a}_n, \hat{b}_0, \cdots \hat{b}_m, \hat{c}_0, \cdots \hat{c}_r] \tag{3.4}$$

$$\varphi^T(k-d) = [y(k-1),...y(k-n),..., u(k-d),..u(k-d-m), \Delta_1, \cdots, \Delta_r] \tag{3.5}$$

$$\varphi_0{}^T(k-d) = [y(k-1),...y(k-n),..., u(k-d),..u(k-d-m), \omega(k-1), \cdots, \omega(k-r)] \tag{3.6}$$

where $\theta_0{}^T$ is the vector form of plant parameters, $\hat{\theta}^T(k)$ is the vector of estimated model parameters, and $\varphi^T(k-d)$ is the regression vector, in which Δ_i depends on the estimation algorithm. In different cases, $\varphi_0{}^T(k-d)$ and $\varphi^T(k-d)$ may be the same or different. With the notation we can write the STC as follows. The output of the plant is given by

$$y(k) = \varphi_0{}^T(k-d)\theta_0 + \omega(k) \tag{3.7}$$

Similarly, the output of the estimated model $\Sigma P_m(k)$ is

$$y'(k) = \varphi^T(k-d)\hat{\theta}(k)$$

The estimates error can be evaluated by

$$\varphi_0{}^T(k-d)\theta_0 - \varphi^T(k-d)\hat{\theta}(k) = y(k) - y'(k) - \omega(k)$$

Let us denote the vector forms of $\Sigma C(k)$ and ΣC by $\theta_c(k)$ and θ_c respectively, then their outputs are

$$u(k) = \varphi_c^T(k)\theta_c(k)$$

$$u'(k) = \varphi_c^T(k)\theta_c(k)$$

where

$$\varphi_c{}^T(k) = [y(k),..., u(k-1), \cdots, \Lambda_1, \cdots,] \tag{3.8}$$

and the elements of $\varphi_c{}^T(k)$ depend on specific controller.

Associated with the STC system described above, we introduce the so-called *virtual equivalent system* as shown in Fig. 3.2, where $\Delta u(k)$ is the difference between outputs of the real controller $\Sigma C(k)$ and the virtual controller ΣC:

$$\Delta u(k) = u(k) - u'(k) = \varphi_c^T(k)\theta_c(k) - \varphi_c^T(k)\theta_c \tag{3.9}$$

From Fig. 3.2 we see that the output of plant ΣP comprises two parts. One part is the output induced by reference input $y_r(k)$ and noise $\omega(k)$, the other is the output caused by $\Delta u(k)$. By "equivalent" we mean that the inputs and the outputs of plant ΣP in the two systems shown in Fig.3.1 and Fig.3.2 are the same. From Fig.3.1 and Fig.3.2 we can see the quantitative difference between an adaptive control system and its corresponding non-adaptive control system as shown in Fig.3.3.

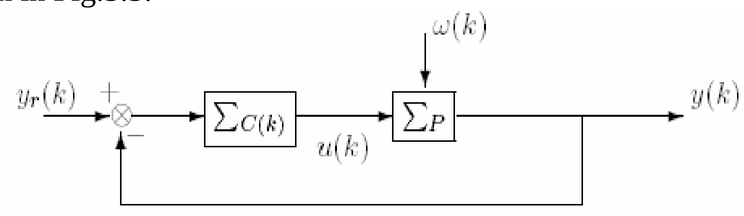

Fig 3.1. STC system of a stochastic plant.

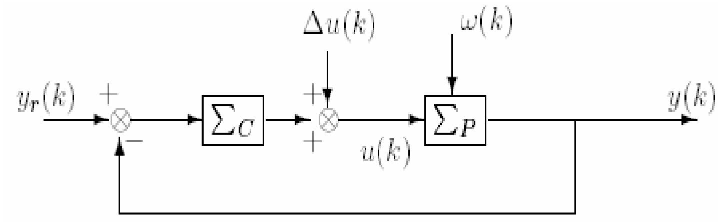

Fig. 3.2. Virtual equivalent system

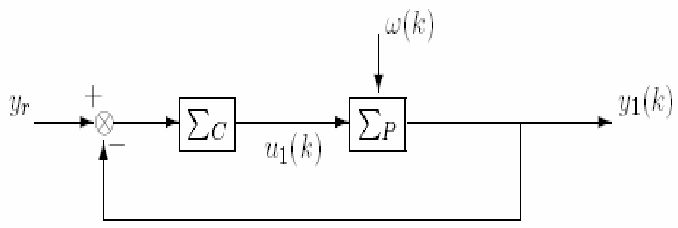

Fig. 3.3. Non-adaptive control system

By stability of the STC system we mean that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(\|y(i)\|^2 + \|u(i)\|^2) < \infty \ \ a.s. \tag{3.10}$$

By convergence of the STC system we mean that the STC system and its corresponding non-adaptive control system have the same input-output index:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\| y(i) - y_r(i)\|^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\| y_1(i) - y_r(i)\|^2 \tag{3.11}$$

3.2 Main Results

The main results are stated as below.

Theorem 3.1:

If the STC system has the following properties:

1) The controller is well defined such that $(\Sigma C, \Sigma P)$ constitutes a stable closed-loop system.

2) The estimator satisfies

$$\sum_{k=1}^{n} \left\| \varphi^T(k-d)\hat{\theta}(k) - \varphi_0{}^T(k-d)\theta_0 \right\|^2 = \sum_{k=1}^{n} \left\| y(k) - y'(k) - \omega(k) \right\|^2 = o(1 + \sum_{k=1}^{n} \left\| \varphi(k-d) \right\|^2) \ a.s.$$

3) The mapping from estimated parameters into controller parameters satisfies

$$\sum_{k=1}^{n} \left\| \varphi_c^T(k)\hat{\theta}_c(k) - \varphi_c^T(k)\theta_c \right\|^2 = O(\sum_{k=1}^{n} \left\| \varphi^T(k-d)\hat{\theta}(k) - \varphi_0{}^T(k-d)\theta_0 \right\|^2) \quad a.s.$$

Then the STC system is stable and convergent.

Corollary:

If the STC system has the following properties:

1) The controller is well defined such that $(\Sigma C, \Sigma P)$ constitutes a stable closed-loop system.

2) The estimator satisfies

$$\sum_{k=1}^{n} \left\| \varphi^T(k-d)\hat{\theta}(k) - \varphi_0{}^T(k-d)\theta_0 \right\|^2 = \sum_{k=1}^{n} \left\| y(k) - y'(k) - \omega(k) \right\|^2 = o(n) \ a.s.$$

3) The mapping from estimated parameters into controller parameters satisfies

$$\sum_{k=1}^{n} \left\| \varphi_c^T(k)\hat{\theta}_c(k) - \varphi_c^T(k)\theta_c \right\|^2 = O(\sum_{k=1}^{n} \left\| \varphi^T(k-d)\hat{\theta}(k) - \varphi_0{}^T(k-d)\theta_0 \right\|^2) \quad a.s.$$

Then the STC system is stable and convergent.

3.3 Proof of Theorem

First, from conditions 2) and 3) we have

$$\frac{1}{n}\sum_{k=1}^{n} \left\| \Delta u(k) \right\|^2 = o(\frac{1}{n}\sum_{k=1}^{n} \left\| \varphi(k-d) \right\|^2) \ a.s. \tag{3.12}$$

Now we decompose the virtual equivalent system into two subsystems as shown in Fig.3.4 and Fig.3.5.

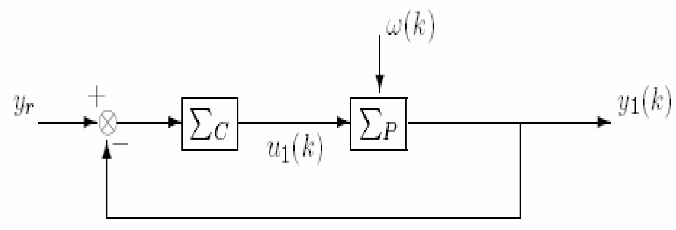

Fig. 3.4. Subsystem 1

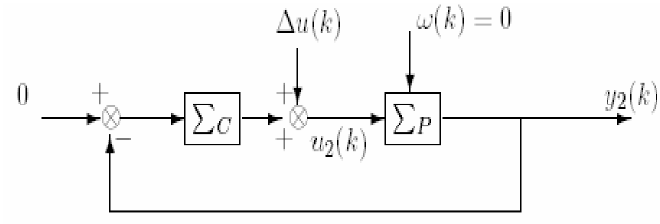

Fig.3.5. Subsystem 2

By superposition property, we have

$$y(k) = y_1(k) + y_2(k) \tag{3.13}$$

$$u(k) = u_1(k) + u_2(k) \tag{3.14}$$

First, we consider subsystem 1 (Fig.3.4). Since it is a time-invariant stochastic system and condition 1) guarantees the closed-loop system to be stable, we have

$$\frac{1}{n}\sum_{i=1}^{n}\left\|\varphi_1(k-d)\right\|^2 < \infty \quad a.s. \tag{3.15}$$

$$\lim_{n\to\infty}\sup\frac{1}{n}\sum_{i=1}^{n}(\left\|y_1(i)\right\|^2 + \left\|u_1(i)\right\|^2) < \infty \quad a.s. \tag{3.16}$$

Next, we analyze subsystem 2 (Fig. 3.5). Since it is a closed-loop stable system, we have [3]

$$\sum_{k=1}^{n} \| y_2(k) \|^2 \le M_1 \sum_{k=1}^{n} \left\|\Delta u(k)\right\|^2 + M_2,\ 0 < M_1 < \infty,\ 0 \le M_2 < \infty,$$

$$\sum_{k=1}^{n} \| u_2(k) \|^2 \le M_3 \sum_{k=1}^{n} \left\|\Delta u(k)\right\|^2 + M_4,\ 0 < M_3 < \infty,\ 0 \le M_4 < \infty,$$

which in turn yields

$$\frac{1}{n}\sum_{k=1}^{n} \| y_2(k) \|^2 = O(\frac{1}{n}\sum_{k=1}^{n}\left\|\Delta u(k)\right\|^2) \quad a.s. \tag{3.17}$$

$$\frac{1}{n}\sum_{k=1}^{n} \| u_2(k) \|^2 = O(\frac{1}{n}\sum_{k=1}^{n}\left\|\Delta u(k)\right\|^2 \quad a.s. \tag{3.18}$$

So by (3.12), we have

$$\frac{1}{n}\sum_{k=1}^{n} \| y_2(k) \|^2 = o(\frac{1}{n}\sum_{k=1}^{n} \|\varphi(k-d)\|^2) \quad a.s. \tag{3.19}$$

$$\frac{1}{n}\sum_{k=1}^{n} \| u_2(k) \|^2 = o(\frac{1}{n}\sum_{k=1}^{n} \|\varphi(k-d)\|^2) \quad \text{a.s.} \tag{3.20}$$

Thus,

$$\frac{1}{n}\sum_{i=1}^{n} \|\varphi_2(k-d)\|^2 = o(\frac{1}{n}\sum_{k=1}^{n} \|\varphi(k-d)\|^2$$

From (3.13) and (3.14), we see that

$$\varphi(k-d) = \varphi_1(k-d) + \varphi_2(k-d)$$

Therefore,

$$\frac{1}{n}\sum_{i=1}^{n} \|\varphi(k-d)\|^2 = \frac{1}{n}\sum_{i=1}^{n} \|\varphi_1(k-d) + \varphi_2(k-d)\|^2$$

$$\leq \frac{1}{n}\sum_{i=1}^{n} \|\varphi_1(k-d)\|^2 + \frac{1}{n}\sum_{i=1}^{n} \|\varphi_2(k-d)\|^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} \|\varphi_1(k-d)\|^2 + o(\frac{1}{n}\sum_{k=1}^{n} \|\varphi(k-d)\|^2) \tag{3.21}$$

Also, from (3.16) we can get

$$\frac{1}{n}\sum_{i=1}^{n} \|\varphi_1(k-d)\|^2 < \infty \tag{3.22}$$

From this together with (3.22), we can obtain by contradiction argument that

$$\frac{1}{n}\sum_{i=1}^{n} \|\varphi(k-d)\|^2 < \infty \quad a.s. \tag{3.23}$$

It thus follows that

$$\lim_{n\to\infty}\sup \frac{1}{n}\sum_{i=1}^{n} (\|y(i)\|^2 + \|u(i)\|^2) < \infty \quad a.s.$$

This shows the stability of the STC system.
Now we prove the convergence of the STC system. First, noticing that by (3.19), (3.20) and (3.23), we can get

$$\frac{1}{n}\sum_{k=1}^{n} \| y_2(k) \|^2 = o(1) \tag{3.24}$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(y(i)-y_r(i))^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}[(y_1(i)-y_r(i))+y_2(i)]^2$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}[(y_1(i)-y_r(i)]^2+\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}[(y_2(i)]^2+\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}[(y_1(i)-y_r(i)]y_2(i)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}[(y_1(i)-y_r(i)]^2+\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}[(y_1(i)-y_r(i)]y_2(i)$$

To estimate the second term we make use of Cauchy's inequality and obtain that

$$0\le\left\{\frac{1}{n}\sum_{i=1}^{n}[(y_1(i)-y_r(i)]y_2(i)\right\}^2\le\{\frac{1}{n}\sum_{i=1}^{n}[(y_1(i)-y_r(i)]^2\}.\{\frac{1}{n}\sum_{i=1}^{n}[(y_2(i)]^2\}$$

By (3.24), the right-hand side tends to 0 as $n\to\infty$. Thus,

$$\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=1}^{n}[(y_1(i)-y_r(i)]y_2(i)\right\}^2=0$$

So we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}[(y_1(i)-y_r(i)]y_2(i)=0$$

From this we finally conclude that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\| y(i)-y_r(i)\|^2=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\| y_1(i)-y_r(i)\|^2$$

This completes the proof of the theorem 3.1 .

IV. Discussion

In this section, we demonstrate how to derive the well-known results of [3] [4] in a unified manner in the presented framework. Note that $y^*(k)$ and $y_r(k)$ are the same thing.

First, we consider the STC system studied in Goodwin, Ramadge and Caines [3]. The estimation algorithm is based on the stochastic gradient (SG) method:

$$\varphi^T(k-1)=[y(k-1),...y(k-n),...,u(k-1),..u(k-1-m),y^*(k-1),\cdots,y^*(k-r)]$$

$u(k)$ is defined from

$$\theta^T(k)\varphi(k) = y^*(k+1)$$

Accordingly, subsystem 1 (shown in Fig.3.4), i.e. $(\Sigma C, \Sigma P)$ yields a closed–loop system

$$y(k+1) = y^*(k+1) + \omega(k+1)$$

It is obviously stable. Then condition 1) of Theorem 3.1 is satisfied.

From [3 , eq. (7.17)] we have

$$\sum_{k=1}^{n} z_i(k)^2 = o(r_n)\,,\ a.s.$$

$$r_n = 1 + \sum_{k=0}^{n} \|\varphi(k)\|^2$$

$$z_i(k) = y_i(k+1) - \varphi^T(k)\hat{\theta}(k) - v(k+1)$$

$$y_i(k+1) = \varphi^T(k)\hat{\theta}(k) + v(k+1)$$

So,

$$z_i(k) = \varphi^T(k)\theta_0 - \varphi^T(k)\hat{\theta}(k)$$

And

$$\sum_{k=1}^{n} z_i(k)^2 = o(r_n)$$

means that

$$\sum_{k=1}^{n} \left\|\varphi^T(k-d)\hat{\theta}(k) - \varphi_0{}^T(k-d)\theta_0\right\|^2 = o(1 + \sum_{k=1}^{n} \|\varphi(k-d)\|^2)$$

Hence, condition 2) of Theorem 3.1 is satisfied. Also, since

$$B_1 u'(k) - \hat{B}_1 u(k) = B_1 u'(k) - B_1 u(k) + (B_1 - \hat{B}_1) u(k)$$

we get

$$\Delta u(k) = B^{-1}{}_1(k)[\varphi^T(k)(\hat{\theta}(k) - \theta_0)]$$

This means that condition 3) of Theorem 3.1 is satisfied. Then we conclude that the STC system described in [3] is stable and convergent.

Second, we consider the STC system studied in Guo and Chen [4]. In this case, it takes $d=1$ for ΣP and the estimation algorithm is the extended least-square (ELS) method:

$$\varphi^T(k-1)=[y(k-1),...y(k-n),...,u(k-1),..u(k-1-m),\hat{\omega}(k-1),\cdots,\hat{\omega}(k-r)]$$

$$\hat{\omega}(k)=y(k)-\hat{\theta}^T(k)\varphi(k-1)$$

$u(k)$ is defined from

$$\theta^T(k)\varphi(k)=y^*(k+1)$$

Accordingly, subsystem 1 (shown in Fig.3.4), i.e. $(\Sigma C,\Sigma P)$ yields a closed–loop system

$$y(k+1)=y^*(k+1)+\omega(k+1)$$

It is obviously stable. Then condition 1) of Theorem 3.1 is satisfied.

From [4, eq. (4.2)] we have

$$\sum_{k=1}^{n+1}\|\hat{\omega}(k)-\omega(k)\|^2=O\log(r_n)\text{ , }a.s.$$

$$r_n=e+\sum_{k=0}^{n}\|\varphi(k)\|^2$$

From [4, (4.9)] we have

$$\sum_{k=0}^{n}\left\|\varphi^T(k)(\hat{\theta}(k)-\theta_0))\right\|^2=O(r_n^{\delta})\text{ , }a.s.\ \forall\delta\in\left(\frac{2}{\beta},1\right),\ \beta>2$$

Then

$$\sum_{k=1}^{n+1}\|\hat{\omega}(k)-\omega(k)\|^2=o(r_n)=o(1+\sum_{k=1}^{n}\|\varphi(k-1)\|^2)$$

$$\sum_{k=0}^{n}\left\|\varphi^T(k)(\hat{\theta}(k)-\theta_0))\right\|^2=o(r_n)=o(1+\sum_{k=1}^{n}\|\varphi(k-1)\|^2)$$

So, condition 2) of Theorem 3.1 is satisfied. Also, it is not difficult to see that

$$B_1u'(k)-\hat{B}_1u(k)=B_1u'(k)-B_1u(k)+(B_1-\hat{B}_1)u(k)$$

$$\Delta u(k)=B^{-1}{}_1(k)[\varphi^T(k)(\hat{\theta}(k)-\theta_0)]$$

Thus, condition 3) of Theorem 3.1 is satisfied. Therefore, we can conclude that the STC system described in [4] is stable and convergent.

5. Concluding Remarks

This paper presents a unified approach to the analysis of self-tuning control by introducing a new concept of virtual equivalent system and making use of a compensation signal, which is the difference between outputs of the real controller and the virtual controller. This unified theory framework can dramatically simplify the analysis of many adaptive control systems.

We may conclude that self-tuning is a generic character that can be achieved by many adaptive schemes. The study in this paper is only a preliminary step towards a general theory of adaptive control. Potential applications of the approach to related problems such as model reference adaptive control, learning control and multi-model adaptive control remains to be explored.

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Weicun Zhang received the PhD in Control Theory from Tsinghua University (Beijing) in 1993. He has been a visiting research fellow at the University of Michigan from 1997 to 1998, and is currently an associate professor of University of Science and Technology Beijing. His research interests include adaptive control, decision support system and electric power market.

Tianguang Chu received the PhD in Dynamics and Control from Tsinghua University (Beijing) in 1993. He was a visiting research fellow at the University of Melbourne in 2001, and is currently an associate professor of Peking University. His research interests include nonlinear systems, networked control systems, and collective behavior and coordinate control of multi-agent systems.

Long Wang was born in Xian, China on Feb. 13, 1964. He received his Bachelor, Master, and Doctor's degrees in Dynamics and Control from Tsinghua University and Peking University in 1986, 1989, and 1992, respectively. He has held research positions at the University of Toronto, Canada, and the German Aerospace Center, Munich, Germany. He is currently Cheung Kong Chair Professor of Dynamics and Control and Director of Center for Systems and Control of Peking University. His research interests are in the fields of networked systems, hybrid systems, swarm dynamics, cognitive science, collective intelligence, and bio-mimetic robotics.