

Invariant Approximations and Disturbance Attenuation for Constrained Linear Discrete-Time Systems

Yongqiang Wang^{*}, Shanbin Li^{*}, Yongyan Cao^{*}, Youxian Sun^{*}, and Jianxia Shou[†]

^{*}National Laboratory of Industrial Control Technology,
Institute of Modern Control Engineering,
Zhejiang University, Hangzhou, 310027

[†]College of Software Engineering
Zhejiang University of Technology, Hangzhou 310014, China

yqwang@ipc.zju.edu.cn

Abstract

This paper presents a new method to the invariant approximations and disturbance attenuation for constrained Linear Discrete-Time Systems. A novel condition in the form of LMIs is derived to guarantee the asymptotically stable of the system, based on which an optimization problem is constructed to compute the approximation of an invariant set contained in the domain of attraction. This new method contains more free parameters, which allow more freedom when we search the optimal solution and thus reduce the conservatism. Furthermore, a local L_2 gain, which can be used to measure the disturbance attenuation ability, is also computed. Finally numerical examples are presented to show the effectiveness of the proposed method.

Keywords: Invariant set, Disturbance attenuation, Lyapunov function, L_2 gain

I. Introduction

This note concerns the stability analysis and disturbance attenuation for a general discrete-time system subject to input saturation and disturbance. Due to the practical significance and the theoretical challenges, problems for systems subject to saturation have attracted tremendous attention in recent years [1-7, 11, 13]. With the absolute stability analysis tools, such as the circle and Popov criteria, various methods have been developed to estimate the domain of attraction [8, 9]. Theory of set invariance plays a fundamental role in the control of constrained systems. Invariant set can be used to estimate the domain of attraction and much effort has been spent on the enlarging of the invariant set [3, 4].

As for the constrained discrete-time systems, many results have appeared in the recent years. A new condition for an ellipsoid to be invariant was presented in [5] for discrete-time systems subject to actuator saturation, which is shown to be less conservative than the traditional circle criterion. Then in [12] the quadratic Lyapunov function approach was extended and a saturation-dependent Lyapunov function was developed. Fang et.al has developed a new method to do the disturbance rejection for systems subject to actuator saturation [15].

Much attention has also been drawn on the disturbance attenuation for systems subject to actuator saturation [5, 8, 9, 16, 18, 19]. In [16], a multi-objective optimization approach was adopted and the

sufficient conditions for the feasibility of a high-gain controller were derived. The work of [9] gave the definition of disturbance rejection and the L_2 gain bound, and these problems were formulated into LMI (Linear Matrix Inequality) optimization problems by circle and Popov criteria. Two ellipsoids were used to cope with the disturbance rejection problem in [5].

In this paper, we are interested in the estimation of domain of attraction and a new condition is derived to make the estimation as large as possible. A saturation-dependent Lyapunov function is introduced, which incorporates the information of the severity of saturation and thus reduces the conservatism. Furthermore, an original method is constructed to estimate the domain of attraction, which has more extra free parameters than other methods and thus reduces the conservatism. Moreover, in the presence of disturbance, a sufficient condition for a set to be invariant is derived. Then based on this condition, we use the local L_2 gain to measure the disturbance rejection ability for the discrete system and an optimization problem in the form of LMIs is constructed to solve this problem.

The paper is organized as follows. Section 2 gives the problem statement and preliminary lemmas. Section 3 introduces the saturation-dependent Lyapunov function and derives an original optimization problem to estimate the domain of attraction. In Section 4, the disturbance rejection problem is addressed. Numerical examples will be presented in Section 5 to illustrate our approach, and the paper is concluded in Section 6.

Notations: The following notations will be used throughout the paper. \mathbb{R}^n denotes the n dimensional Euclidean space with vector norm $\|\bullet\|$, and $\mathbb{R}^{m \times n}$ denotes the set of all $m \times n$ real matrices. The notation $M > 0$ is used to denote a symmetric positive definite matrix. $\partial\Omega(P, \rho)$ denotes the boundary of $\Omega(P, \rho)$. $\bar{\sigma}$ denotes the maximal singular value of a matrix. $l_2^m([0, N])$ denotes the space of square summable vector sequence over $[0, N]$, i.e., space formed by the sequence $x = \{x_0, x_1, \dots, x_N\}$ with

$$x_k \in \mathbb{R}^m \text{ and such that } \|x\|_2 = \left(\sum_{k=0}^N x_k^T x_k \right)^{1/2} < \infty.$$

II. Preliminaries

Consider the following system subject to actuator saturation

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B\sigma(u(k)), \\ z(k) &= Cx(k), \end{aligned} \tag{1}$$

and $x \in \mathbb{R}^n$ denotes the state vector, $u \in \mathbb{R}^m$ the control input vector and $A \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times l}$, $B \in \mathbb{R}^{n \times m}$ are real-valued matrices. Without loss of generality, we assume that the bounded disturbance w belongs to the set $W := \{w : w(k)^T w(k) \leq 1, \forall k \geq 0\}$.

The function $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the standard saturation function defined by:

$$\sigma(u) = [\sigma(u_1) \quad \sigma(u_2) \quad \dots \quad \sigma(u_m)]^T$$

where $\sigma(u_i) = \text{sign}(u_i) \min\{1, |u_i|\}$.

Consider the following linear state feedback law $u(k) = Fx(k)$. We would like to know how the closed-loop system behaves in the presence of saturation nonlinearity, in particular, to what extent the stability is preserved. In the first step, we aim at obtaining an estimate of the domain of attraction of the origin of the closed-loop system

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B\sigma(Fx(k)), \\ z(k) &= Cx(k), \end{aligned} \tag{2}$$

Let f_i be the i th row of the matrix F . We define the symmetric polyhedron $L(F) = \{x \in \mathbb{R}^n : |f_i x| \leq 1, i = 1, 2, \dots, m\}$. For $x(0) = x_0 \in \mathbb{R}^n$, denote the state trajectory of the systems (2) as

$\varphi(k, x_0, w)$ at time k . A set is said to be invariant if all the trajectories starting from it will remain in it regardless of $w \in W$.

Let $P \in \mathbb{R}^{n \times n}$ be a positive-definite matrix. For a number $\rho > 0$, and ellipsoid $\Omega = (P, \rho)$ is defined as $\Omega(P, \rho) = \{x \in \mathbb{R}^n : x^T P x \leq \rho\}$.

Now we will introduce several important lemmas. Let \mathbb{V} be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in \mathbb{V} . Suppose that each element of \mathbb{V} is labeled as $E_i, i = 1, 2, \dots, 2^m$, and denotes $E_i^- = I - E_i$. Clearly E_i^- is also an element of \mathbb{V} if $E_i \in \mathbb{V}$.

Lemma 1 [5] *Let $F, H \in \mathbb{R}^{m \times n}$ be given. For $x \in \mathbb{R}^n$, if $x \in \mathbb{L}(H)$, then*

$$\sigma(Fx) \in \text{co}\{E_i F x + E_i^- H x : i \in [1, 2^m]\}$$

where $\text{co}\{\cdot\}$ denotes the convex hull of a set. Consequently, $\sigma(Fx)$ can be expressed as

$$\sigma(Fx) = \sum_{i=1}^{2^m} \eta_i (E_i F + E_i^- H) x \quad (3)$$

where η_i is a parameter dependent on the severity of saturation and satisfies $\sum_{i=1}^{2^m} \eta_i = 1, 0 \leq \eta_i \leq 1$.

Note that one of the main advantages of the above lemma is that $\sigma(Fx(k))$ can be represented as a convex hull of a group of linear feedbacks, which will be seen in the following sections.

Lemma 2 [10] *Let $x \in \mathbb{R}^n, H \in \mathbb{R}^{m \times n}$ and assume that $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix, such that $\text{rank} H = r < n$. The following statements are equivalent :*

1. $x^T P x < 0, \forall H x = 0, x \neq 0$
2. $\exists X \in \mathbb{R}^{n \times m} : P + X H + H^T X^T < 0$.

III. Stability Analysis by a Saturation-Dependent Lyapunov Function

In this section, we will use a newly presented saturation-dependent Lyapunov function in [12] to analyze the stability of the saturated system (2) by the invariant set approach.

To clearly present the problem, we denote $A_i = A + B(E_i F + E_i^- H)$, where $H \in \mathbb{R}^{m \times n}$ satisfies $\|Hx\|_\infty \leq 1$.

Following Lemma 1, we can rewrite the systems (2) as follows

$$x(k+1) = \tilde{A}(\eta(k))x(k) + B_1 w(k), \forall x \in \mathbb{L}(H) \quad (4)$$

$$z(k) = Cx(k),$$

where $\tilde{A}(\eta(k)) := \sum_{i=1}^{2^m} \eta_i(k) A_i$, and $\eta(k) = [\eta_1(k) \ \eta_2(k) \ \dots \ \eta_{2^m}(k)]$ is time-varying parameter dependent on

$x(k)$ and $\sum_{i=1}^{2^m} \eta_i(k) = 1, 0 \leq \eta_i(k) \leq 1$. It is easy to see that parameters $\eta(k)$ depend on the severity of the saturation [12], for example, if all actuators are not saturated at time k , we have $\eta_1(k) = 1, \eta_i(k) = 0$ for $i = 2, 3, \dots, 2^m$. In what follows, we will use $\eta_i(k)$ to denote $\eta_i(x(k))$.

A. Stability analysis

With a positive-definite matrix $P \in \mathbb{R}^{n \times n}$, a quadratic Lyapunov function can be defined as $V(x(k)) = x^T P x$. For $\rho > 0$, a level set of $V(\cdot)$, denoted by $L_v(\rho)$, is $L_v(\rho) := \{x \in \mathbb{R}^n : V(x(k)) \leq \rho\} = \Omega(P, \rho)$.

The unknown but measurable time-varying parameters $\eta(k)$ can provide real-time information on the variations of the saturation. To reduce the conservatism in analyzing the stability of the saturated system (2), it is desirable to use this information on saturation. In what follows, we introduce this new saturation-dependent Lyapunov function [12]:

$$V(x(k)) = x^T(k) P(\eta(x(k))) x(k) \quad (5)$$

where $P(\eta(x(k))) := \sum_{i=1}^{2^m} \eta_i(x(k))P_i$. Then the estimation of the domain of attraction is obtained by the Lyapunov level set approach. Define $\Omega_\eta(P(\eta), \rho) = L_\nu(P, \rho) = \{x \in \mathbb{R}^n : x^T P(\eta(x(k)))x \leq \rho\}$.

Definition 1 The closed-loop system (2) is regional asymptotically stable at the origin with the level set $L_\nu(\rho)$ contained in the domain of attraction if $L_\nu(\rho) \subset L(H)$ for some $H \in \mathbb{R}^{n \times n}$, and

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0$$

for any $x(k) \in L_\nu(\rho) \setminus \{0\}$.

In what follows, a condition under which $\Delta V(x(k)) < 0$ holds will be given with the general $P(\eta(x(k)))$ for closed-loop system (2).

Theorem 1 Consider the closed-loop system (2) with $w \equiv 0$ under a given state feedback control matrix F . If there exist matrices $N_1 \in \mathbb{R}^{n \times n}$, $N_2 \in \mathbb{R}^{n \times n}$, $P_i \in \mathbb{R}^{n \times n}$, and $P_i > 0, i = 1, 2, \dots, 2^m$, such that

$$\begin{bmatrix} N_2 A_i + A_i^T N_2^T - P_i & A_i^T N_1^T - N_2 \\ N_1 A_i - N_2^T & -N_1 - N_1^T + P_j \end{bmatrix} < 0, \quad \forall i, j \in [1, 2^m] \quad (6)$$

and $L_\nu(P, \rho) \subset L(H)$. Then the closed-loop system (2) in absence of disturbance is regional asymptotically stable at the origin with the level set $L_\nu(P, \rho)$ contained in the domain of attraction.

Proof. Choose Lyapunov function (5). As the statement of definition 1, given $L_\nu(P, \rho) \subset L(H)$, the closed-loop system (2) is regional asymptotically stable at the origin, if

$$\begin{aligned} \Delta V(x(k)) &= \tilde{A}^T(\eta(k))P(\eta(k+1))\tilde{A}(\eta(k)) - P(\eta(k)) \\ &= \sum_{j=1}^{2^m} x^T(k+1)\eta_j(k+1)P_j x(k+1) - \sum_{i=1}^{2^m} x^T(k)\eta_i(k)P_i x(k) \\ &= \sum_{i=1}^{2^m} \sum_{j=1}^{2^m} \eta_i(k)\eta_j(k+1)(x^T(k+1)P_j x(k+1) - x^T(k)P_i x(k)) < 0 \end{aligned}$$

for any $x(k) \in L_\nu(P, \rho) \setminus \{0\}$.

It is easy to see that

$$\begin{cases} \begin{bmatrix} x^T(k) & x^T(k+1) \end{bmatrix} \begin{bmatrix} -P_i & 0 \\ 0 & P_j \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix} < 0, \quad \forall i, j \in [1, 2^m] \\ \begin{bmatrix} \sum_{i=1}^{2^m} \eta_i(k)A_i & -I \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix} = 0, \quad \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix} \neq 0 \end{cases} \quad (7)$$

is sufficient for $\Delta V(x(k)) < 0$.

Let $H = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, and $N_1, N_2 \in \mathbb{R}^{n \times n}$. By lemma 2, (7) is equivalent to

$$\begin{aligned} &\begin{bmatrix} \sum_{i=1}^{2^m} \eta_i(k)(N_2 A_i + A_i^T N_2^T) - P_i & \sum_{i=1}^{2^m} \eta_i(k)A_i^T N_1^T - N_2 \\ \sum_{i=1}^{2^m} \eta_i(k)N_1 A_i - N_2^T & -N_1 - N_1^T + P_j \end{bmatrix} \\ &= \sum_{i=1}^{2^m} \eta_i(k) \begin{bmatrix} N_2 A_i + A_i^T N_2^T - P_i & A_i^T N_1^T - N_2 \\ N_1 A_i - N_2^T & -N_1 - N_1^T + P_j \end{bmatrix} < 0, \quad \forall i, j \in [1, 2^m] \end{aligned} \quad (8)$$

Obviously, if inequality (6) holds, $\Delta V(x(k)) < 0$. And then we can conclude system (2) is asymptotically stable at the origin with $L_\nu(P, \rho)$ contained in the domain of attraction.

Remark 1 Theorem 1 gives a condition that ensures the Lyapunov level set $L_\nu(P, \rho)$ to be contained in the domain of attraction for the closed system (2) in absence of disturbance under a linear constant feedback control law, while maintaining the stability of the closed-loop system. Actually, if we set $N_2 = 0$, inequality (6) becomes

$$\begin{bmatrix} -P_i & A_i^T N_1^T \\ N_1 A_i & -N_1 - N_1^T + P_j \end{bmatrix} < 0, \quad \forall i, j \in [1, 2^m] \quad (9)$$

Inequality (9) is exactly the same as the Theorem 1 of [12]. Hence, the result of [12] is just a special situation of our paper. Free parameter N_1, N_2 provides more freedom to obtain a less conservative result.

B. Estimation of Domain of Attraction

Actually, a sufficient condition for a level set to be invariant has been obtained in Theorem 1. A larger estimation of domain of attraction is more desirable in the stability analysis for systems. Hence it is natural for us to choose the largest level set to estimate the domain of attraction. A shape reference set, in terms of a polyhedron or ellipsoid, is always adopted to measure the size of the domain of attraction [5]. Let $X_R \subset \mathbb{R}^n$ be a prescribed bounded convex set containing the origin. For a set $L \subset \mathbb{R}^n$ which contains the origin, define $\beta(X_R, L) := \sup\{\beta > 0 : \beta X_R \subset L\}$. Here we choose X_R to be a polyhedron defined as $X_R = \text{co}\{x_1, x_2, \dots, x_l\}$.

With the above shape reference sets, we can choose from all the $L_v(P, \rho)$'s that satisfy the condition of Theorem 1 such that the quality β is maximized. This problem can be formulated as the following constrained optimization problems:

OPT1: $\max_{\rho > 0, N_1, N_2, H} \beta, s.t.$ (a) $\beta X_R \subset L_v(P, \rho)$, (b) inequality (6), and (c) $|h_i x| \leq 1, \forall x \in L_v(P, \rho)$, where h_i denotes the i th row of H .

As in [12], we use $\bigcap_{i=1}^{2^m} \Omega(P_i, \rho)$ to estimate $L_v(P, \rho)$. Hence OPT1 can be solved by the following optimization problem.

OPT2: $\max_{\rho > 0, N_1, N_2, H} \beta, s.t.$ (a) $\beta X_R \subset \Omega(P_i, \rho)$, (b) inequality (6), and (c) $|h_i x| \leq 1, \forall x \in \Omega(P_i, \rho)$.

Without loss of generality, we will let $\rho = 1$ in the left part of this section. To solve problem OPT2, let $N_2 = \delta N_1$, $X = N_1^{-T}$, and $Q_i = X^T P_i X$. Note that $X^T Q_i^{-1} X \geq X^T + X - Q_i$. With the given shape reference set X_R , constraint (a) is equivalent to

$$\beta^2 x_j^T P_i x_j \leq 1 \Leftrightarrow \begin{bmatrix} \beta^{-2} & x_j^T \\ x_j & X(X^T P_i X)^{-1} X^T \end{bmatrix} \geq 0 \Leftrightarrow \begin{bmatrix} \beta^{-2} & x_j^T \\ x_j & X^T + X - Q_i \end{bmatrix} \geq 0 \quad (10)$$

Pre and post-multiplying inequality (6) by $\text{diag}\{X^T, X^T\}$ and $\text{diag}\{X, X\}$ respectively. Then it is clear that constraint (b) is equivalent to

$$\begin{bmatrix} \delta A_i X + \delta X^T A_i^T - Q_i & * \\ AX + B(E_i F X + E_i^- Z) - \delta X^T & -X - X^T + Q_j \end{bmatrix} < 0, \quad \forall i, j \in [1, 2^m] \quad (11)$$

where $Z = HX$. Let $z_j = h_j X$. Constraint (c) is equivalent to

$$h_j P_i^{-1} h_j^T \leq 1 \Leftrightarrow 1 - h_j X (X^T P_i X)^{-1} X^T h_j^T \geq 0 \Leftrightarrow \begin{bmatrix} 1 & h_j X \\ X^T h_j^T & Q_i \end{bmatrix} \geq 0 \Leftrightarrow \begin{bmatrix} 1 & z_j \\ z_j^T & Q_i \end{bmatrix} \geq 0 \quad \forall i, j \in [1, 2^m] \quad (12)$$

Based on the description above, the problem of enlarging the domain of attraction can be reduced to an LMI optimization problem defined as follows.

$$\begin{aligned} \text{OPT3} \quad & \min_{\rho > 0, X, Z} \mathcal{G}, \quad s.t. \\ & (a) \begin{bmatrix} \mathcal{G} & x_j^T \\ x_j & X^T + X - Q_i \end{bmatrix} \geq 0 \quad \forall j \in [1, l], i \in [1, 2^m], \\ & (b) \text{Inequalities (11), (12)} \end{aligned} \quad (13)$$

where $\mathcal{G} = \beta^{-2}$.

IV. Disturbance Attenuation

In this section, we will do set invariance and Local L_2 gain analysis for system (2). Before proceeding to the main result, definition concerning the disturbance rejection is given as what follows:

Definition 2 (γ -disturbance attenuation): *Given an invariant set Ω_{in} , the closed-loop system (2) is said to be regional stable at the origin with γ -disturbance attenuation if, for all $w \in W$ and $x_k \in \Omega_{in}$, the origin of the closed-loop system is asymptotically stable and the response $\{z_k\}$ of the system under the zero initial condition satisfies*

$$\|z\|_2 < \gamma \|w\|_2 \quad (14)$$

Consider the system (2). For all $w \in W$, one principal concern is how far the trajectories will extend from the origin. Here we will use an ellipsoid to bound the trajectories of the system. We say that the ellipsoid $\Omega(P, \rho)$ is strictly invariant if

$$(Ax(k) + B_1w(k) + B\sigma(u(k)))^T P(\eta(k+1))(Ax(k) + B_1w(k) + B\sigma(u(k))) < \rho$$

for all $x(k) \in \Omega(P, \rho)$ and $w(k) \in W$.

Theorem 2 *Consider system (2) and assume $L_v(P, \rho) \subset L(H)$. Given linear feedback law F , $L_v(P, \rho)$ is an invariant set for system (2), if there exist matrices $S \in \mathbb{R}^{n \times n}$, $P_i \in \mathbb{R}^{n \times n}$, and $P_i > 0, i = 1, 2, \dots, 2^m$, such that*

$$\begin{bmatrix} -gP_i & * \\ SA_i & -S^T - S + P_j \end{bmatrix} \leq 0, \quad \forall i, j \in [1, 2^m], \quad (15)$$

where $g = \frac{1}{1+\mu} \left(1 - \frac{1+\mu}{\mu} \bar{\sigma}\right)$, and $\bar{\sigma}$ denotes $\bar{\sigma}(B_1^T P B_1)$

Proof. Since $L_v(P, \rho) \subset L(H)$, and by Lemma 1, $\forall x \in L_v(P, \rho)$ system (2) can be rewritten as system (4). Select the Lyapunov function (5). By theorem 2 in [5], the invariance of $L_v(P, \rho)$ can be obtained if there exists $\mu > 0$, such

$$(1+\mu)(\tilde{A}(\eta(k)))^T P(\eta(k+1))(\tilde{A}(\eta(k))) + \left(\frac{1+\mu}{\mu} \bar{\sigma}(B_1^T P(\eta(k+1))B_1) - 1\right) P(\eta(k)) \leq 0$$

To simply the problem, let $\bar{\sigma}$ denote $\bar{\sigma}(B_1^T P(\eta(k+1))B_1)$. By Schur complement, the above is equivalent to

$$\begin{bmatrix} \frac{1}{1+\mu} \left(1 - \frac{1+\mu}{\mu} \bar{\sigma}\right) P(\eta(k)) & * \\ \tilde{A}(\eta(k)) & -P^{-1}(\eta(k+1)) \end{bmatrix} \leq 0 \quad (16)$$

Let $g = \frac{1}{1+\mu} \left(1 - \frac{1+\mu}{\mu} \bar{\sigma}\right)$. Noting that inequality $-S^T P S \leq -S^T - S + P^{-1}$ and pre and post-multiplying (16)

by $\text{diag}\{I, S\}$ and $\text{diag}\{I, S^T\}$ respectively. It is sufficient for inequality (16) to hold if

$$\begin{bmatrix} g \left(-\sum_{i=1}^{2^m} \eta_i(k) P_i\right) & * \\ S\tilde{A} & -S^T - S + \left(\sum_{j=1}^{2^m} \eta_j(k+1) P_j\right) \end{bmatrix} \leq 0 \quad (17)$$

$$= \sum_{i=1}^{2^m} \sum_{j=1}^{2^m} \eta_i(k) \eta_j(k+1) \begin{bmatrix} -gP_i & * \\ SA_i & -S^T - S + P_j \end{bmatrix} \leq 0, \quad \forall i, j \in [1, 2^m]$$

Obviously inequality (15) is sufficient for inequality (17) to hold. This complete the proof.

By Theorem 2, we can obtain an invariant set $L_v(P, \rho)$ to bound all the state trajectories. In what follows, we will use invariant set to deal with disturbance rejection for the closed-loop system. L_2 gain is always used to measure the disturbance attenuation capability. For a linear system subject to actuator saturation, we can not get the global L_2 gain in the common sense. However, a local L_2 gain can be obtained. That is to say we can use an invariant set to bound the state trajectories

of the system for all $w(k) \in W$ and then we can estimate the upper bound on the L_2 gain for a closed-loop system.

Theorem 3 *The system (2) is regional stable at the origin with γ -disturbance attenuation, i.e., $\|z\|_2 < \gamma \|w\|_2$ in invariant set $L_\nu(P, \rho)$ for all the nonzero disturbance $w(k) \in W$ under the zero initial condition, if in addition to (15), there exist matrices $S, M \in \mathbb{R}^{n \times n}, P_i \in \mathbb{R}^{n \times n}, P_i > 0, \forall i \in [1, 2^m]$ and scalar $\gamma > 0$, such that*

$$\Gamma = \begin{bmatrix} P_j - S - S^T & * & * \\ A_i^T S - M^T & -P_i + C^T C + A_i^T M + M^T A_i & * \\ B_i^T S & B_i^T M & -\gamma^2 I \end{bmatrix} < 0 \quad (18)$$

and $L_\nu(P, \rho) \subset L(H)$.

Proof. Note that inequality (18) implies (6), so the closed-loop system (2) with $w(k) = 0$ is regional asymptotically stable. Consider the system (2). Select the Lyapunov function (5). By Lemma 1, system (2) can be rewritten as system (4).

By Theorem 2, inequalities (15) ensure that $L_\nu(P, \rho)$ is an invariant set. Define $J_N \square \sum_{k=0}^{N-1} (z_k^T z_k - \gamma^2 w_k^T w_k)$.

Without loss of generality, the initial condition is assumed to be zero. Define $\bar{x}_k = [x_{k+1} \quad x_k \quad w_k]^T$.

Then we obtain

$$J_N = \sum_{k=0}^{N-1} (z_k^T z_k + V(x(k+1)) - V(x(k)) - \gamma^2 w_k^T w_k) - V(x_N) \leq \sum_{k=0}^{N-1} \bar{x}_k^T \Xi_0 \bar{x}_k \quad (19)$$

Where

$$\Xi_0 = \begin{bmatrix} P(\eta(k+1)) & 0 & 0 \\ * & -P(\eta(k)) + C^T C & 0 \\ * & * & -\gamma^2 I \end{bmatrix}. \quad (20)$$

Note that $\Xi_0 < 0$ if the following is true

$$\begin{cases} \bar{x}_k^T \Xi_0 \bar{x}_k < 0 \\ \begin{bmatrix} -I & \tilde{A} & B_1 \end{bmatrix} \bar{x}_k & \bar{x}_k \neq 0 \end{cases} \quad (21)$$

Let $X = \begin{bmatrix} S \\ M \\ 0 \end{bmatrix}$. By Lemma 2, (21) is equivalent to

$$\begin{bmatrix} P(\eta(k+1)) - S - S^T & * & * \\ \tilde{A}^T S^T - M & -P(\eta(k)) + C^T C + \tilde{A}^T M^T + M \tilde{A} & * \\ B_1^T S^T & B_1^T M^T & -\gamma^2 I \end{bmatrix} = \sum_{i=1}^{2^m} \sum_{j=1}^{2^m} \eta_i(k) \eta_j(k+1) \Gamma < 0 \quad (22)$$

Obviously, if (18) holds, the inequality (22) holds. Hence $J_N < 0$. That is to say $\|z\|_2 < \gamma \|w\|_2$. This completes the proof.

Based on the Theorem 3, we can construct an optimization problem to determine the possible smallest L_2 gain of the system (2) in some invariant set $L_\nu(P, \rho)$. However due to the parameter H , inequalities (15), (18) are not LMIs. What follows is presented to make the L_2 gain problem solvable.

Let $X = S^{-T}, Q_i = X^T P_i X$. Pre and post-multiplying inequality (15) by $\text{diag}\{X^T, X^T\}$ and $\text{diag}\{X, X\}$ respectively. Then inequality (15) is equivalent to

$$\begin{cases} \begin{bmatrix} -g' Q_i & * \\ AX + B(E_i F X + E_i^- Z) & -X^T - X + Q_j \end{bmatrix} \leq 0 \\ \begin{bmatrix} \zeta & * \\ B_1 & X^T + X - Q_j \end{bmatrix} \geq 0 \end{cases} \quad (23)$$

Where $Z = HX$ and $g' = \frac{1}{1+\mu} \left(1 - \frac{1+\mu}{\mu} \zeta \right)$.

Let $M = \delta S$. Pre and post-multiplying inequality (18) by $\text{diag}\{X^T, X^T, I\}$ and $\text{diag}\{X, X, I\}$ respectively. Then inequality (18) is equivalent to

$$\begin{bmatrix} Q_j - X - X^T & \Sigma - \delta X & B_1 & 0 \\ * & -Q_i + \delta(\Sigma^T + \Sigma) & \delta B_1 & X^T C^T \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \quad (24)$$

where $\Sigma = AX + B(E_i F X + E_i^- Z)$ with $Z = HX$.

Hence, the following optimization problem can be constructed to compute the smallest L_2 gain.

OPT4: $\max_{Q_i > 0, Z, X} \gamma^2, s.t.$ (23) and (24).

Note that the maximal value of ζ is $(1 - \sqrt{g'})^2$. Hence given δ , **OPT4** can be solved by varying g' from 0 to 1, which can be solved efficiently by [20].

Table 1: β_F for different δ

δ	0	0.01	0.02	0.03	0.04	0.05
β_F	4.5236	4.5263	4.5284	4.5301	4.5305	4.5311

V. Numerical Examples

Example 1 First, we will present an example to illustrate the effectiveness of our new saturation-dependent Lyapunov function in estimation of domain of attraction. Considering the following closed-loop system with [12]

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}; B_1 = \begin{bmatrix} 0.02 \\ 0.03 \end{bmatrix}; C = [1 \ 1].$$

We design the state feedback control law by the DLQR approach with $Q = I$ and $R = 0.1$. For the above system, we obtain the following controller, $F = [-0.6167 \ -1.2703]$.

Firstly, we consider the estimation of domain of attraction for this system. As in [12], we use the shape reference set of the form $X_R = \left\{ \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix} \right\}, \theta \in [0, 2\pi]$. For this example, when $\theta = 0.4\pi$, we have

$\beta_F = 4.5311$ with $\delta = 0.05$ by our method, while $\beta_s = 4.5235$ by the method of [12]. Detailed result can be seen in Table 1, in which we can see that when $\delta = 0$, we obtain the result as in [12]. Obviously our method is better.

Now we consider the disturbance attenuation problem for this system. If we set $\delta = 0$, that is to mean there is no extra free parameter in Theorem 3, we obtain $\gamma = 0.0557$, while $\gamma = 0.0543$ with $\delta = -0.1$. Hence our method provides more freedom to search the better result and conservatism is reduced.

Example 2 To further illustrate the effectiveness of our approach, we consider a more complex third-order system with

$$A = \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1.67 \\ 0.5 \\ 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0.02 \\ 0.03 \\ 0.03 \end{bmatrix}, C = [1 \ 1 \ 1]$$

A DLQR control law with $Q = I$ and $R = 0.1$ is given as $F = [-0.3683 \ -0.8498 \ -0.7963]$.

To estimate the domain of attraction of this system, we use the shape reference set $X_r = \left\{ \begin{bmatrix} 0.7071 \\ 0.7071 \\ 1 \end{bmatrix} \right\}$.

With $\delta = -0.06$, we obtain $\beta_f = 1.2370$, while $\beta_s = 1.2359$ with $\delta = 0$. Now consider the disturbance attenuation problem of this system. We obtain $\gamma = 0.7800$ if we $\delta = 0$, which mean there is no free parameter in Theorem 3. This result can also be obtained by extending the result of [12] in the way of our approach in Section 4. Then if we set $\delta = 0.08$, we obtain $\gamma = 0.7000$, which is much better than the result when there is no extra free parameter.

VI. Conclusion

In this paper, we have considered the problem of analysis and controller synthesis for discrete-time systems in the presence of actuator saturation and disturbance. We use a newly defined saturation-dependent Lyapunov function to estimate the domain of attraction, which is then formulated and solved as a constrained LMI optimization problem by a new technique. Then as for the disturbance rejection problem, the regional L_2 gain is determined by an optimization problem, which can be used to measure the disturbance rejection ability of the system. Our method is better than that of [12], which is demonstrated by numerical examples.

VII. Acknowledgement

This work is supported by the National Natural Science Foundation of China (under Grant No. 60474045), 973 Plan of China (under Grant No.2002CB312200)

References

- [1]. H.J.Sussmann, E.D.Sontag, and Y.Yang. A general result on the stabilization of linear systems using bounded controls, *IEEE Trans. Automat. Contr.*, 39, PP. 2411-2425, 1994
- [2]. L. Zaccarian and A. R. Teel, A common framework for anti-windup, bumpless transfer and reliable designs, *Automatica*, 38, pp. 1735-1744, 1999
- [3]. F. Blanchini, Set invariance in control, *Automatica*, 35(11), pp. 1747-1767
- [4]. F. Blanchini, S. Miani, and M. Sznajer, Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions., *IEEE Trans. Automat. Contr.*, 39(2), pp. 428-433, 1994
- [5]. T. Hu and Z. Lin and B. M. Chen, Analysis and design for discrete-time linear systems subject to actuator saturation, *System & Control Letters*, 45(2), pp. 97-112, 2002
- [6]. T. Hu and Z. Lin, Composite Quadratic Lyapunov Functions for Constrained Control Systems, *IEEE Trans. Automat. Contr.*, 48, pp. 440-450, 2003
- [7]. Y.-Y. Cao and Z. Lin and D.G.Ward. An anti-windup approach to enlarging domain of attraction for linear systems subject to actuator saturation, *IEEE Trans. Automatic Control*, 2002, 47(1): 140~145
- [8]. Paim. C, Tarbouriech. S, Gomes da Silva Jr. J.M, and Castelan E. B., Control design for linear systems with saturating actuators and L_2 -bounded disturbances, *Proceedings of the 41st IEEE conference on Decision and Control*, pp. 4134-4138.2002

- [9]. Hindi. H and Boyd. S, Analysis of Linear Systmes with Saturation using Convex Optimization, *Proceedings of the 37th IEEE conference on Decision and Control*, pp. 903-908.1998
- [10]. M. C. de Oliveira and R. E. Skelton, Stability tests for constrained linear systems, *Perspective in Robust Control, Lecture Notes in Control and Information Science*, pp. 241-257, Springer-Verlag, 2001.
- [11]. Bennett D, Burge SE, Bradshaw A, Design of a controller for a highly coupled V/STOL aircraft, *TRANSACTIONS OF THE INSTITUTE OF MEASUREMENT AND CONTROL*, 21 (2-3), pp. 63-75. 1999
- [12]. Y.-Y. Cao and Z. Lin, Stability analysis of discrete-time systems with actuator saturation by a saturation-dependent Lyapunov function, *Automatica*, 39, pp. 1235-1241, 2003
- [13]. Chen J, Patton RJ, Chen Z, Active fault-tolerant flight control systems design using the linear matrix inequality method, *TRANSACTIONS OF THE INSTITUTE OF MEASUREMENT AND CONTROL*, 21 (2-3): 77-84 1999
- [14]. Y.Q. Wang, Y.-Y. Cao, and Y.X. Sun, Stability analysis and anti-windup design for discrete-time systems by a saturation-dependent Lyapunov function approach, IFAC 2005.
- [15]. Haijun Fang, Zongli Lin, and Tingshu Hu, Analysis of linear systems in the presence of actuator saturation and L_2 -disturbance. *Automatica*, 40, pp. 1229-1238, 2004
- [16]. Nguyen. T and Jabbari. F, Disturbance attenuation for systmes with input saturation: An LMI approach. *IEEE Trans. Automatic Control*, 44(4), pp. 852-857, 1999.
- [17]. A.Benzaouia, L. Saydy, Stability and Control Synthesis of Switched Systems Subject to Actuator Saturation. *Proceeding of the 2004 American Control Conference*, pp. 5818-5823, 2004
- [18]. Duan GR, Wu ZY, Howe D, Robust control of a magnetic-bearing flywheel using dynamical compensators, *TRANSACTIONS OF THE INSTITUTE OF MEASUREMENT AND CONTROL*, 23 (4): 249-278 2001
- [19]. Nguyen. T and Jabbari. F, Output feedback controllers for disturbance attenuation with actuator amplitude and rate saturation. *Automatica*, 36, pp. 1339-1346, 2000.
- [20]. S.Boyd, L.El Ghaoui, E.Feron and V.Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM,Philadelphia, 1994.



Yongqiang Wang Received his bachelor degree from Zhejiang University in 2003. Now he is a Master Candidate of the Department of Control Science and Engineering at Zhejiang University. His research interests include the nonlinear, roust control, constrained system control, and networked control systems.



Shanbin Li was born in Jiangxi, China, in 1978. He received his B.S. degree in Industrial Automation from Nanchang Univeristy, China in 2000. From September 2000, he was a Ph.D. candidate and received the Ph.D. degree in control science and control engineering in 2005 from Department of Control, Zhejiang Univeristy, China. From April 2004 and May 2004, he was a Research Associate at TRIO-LORIA (Laboratoire Lorrain de Recherche en Informatique et ses Applications), France. From July 2005 to August 2005, he was a Research

Associate at the University of Hong Kong. He is now a Post-doctoral Research Fellow at CRAN, Université Henri Poincaré, France. His research interests include networked control systems, stochastic systems, time-delay systems, robust control, fault detection and fault-tolerant control.

Yong-Yan Cao was born in Hunan, China, in 1968. He received the B.E. and M.Sc. degrees in electrical engineering from Wuhan University of Science and Technology, China, and the Ph.D. degree in industrial automation from Zhejiang University, China, in 1990, 1993, and 1996, respectively. From 1997 to 1998, he was a Visiting Research Associate in the Department of Mechanical Engineering, The University of Hong Kong. From 1998 to 1999, he held an Associate Professor position at Zhejiang University. He then spent a year-and-a-half with the Department of Measurement and Control, Duisburg University, Germany, as an Alexander von Humboldt Research Fellow. He has published more than 50 papers in some reputed journals and conferences such as the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, Automatica, and Systems and Control Letters. He also has been active both in his own research and in his service as a Reviewer for many journals and conferences. His current research interests include robust and nonlinear control, constrained control, time-delay systems, sampled-data systems, and fuzzy control.



SUN You-Xian Youxian Sun was born in Zhejiang, China in 1940. He graduated in Chemical Engineering Dept. at Zhejiang Univ. in 1964. In the same year, he became a teacher in the Chemical Engineering Dept. and Institute of Industrial Process Control. From 1984 to 1987, he went to Chemical Engineering Dept. at Univ of Stuttgart, Germany as a visitor. He was promoted as Professor in 1988 and Ph.D. advisor in 1991. He was awarded the model worker in the National Educational System and the Medal of the People Teacher. He was the delegate of the Eighth National Peoples Congress. And in 1995, he was elected as the Academician of the Engineering Academics and the Zhejiang primary commissary of Chinese Democracy Federation in 1996. He was awarded the first National Excellent Science and Technology worker in 1999. In the same year, he was elected the Adjunct Director of the Standing Committee of People's Congress of Zhejiang Province. His research fields include robust control theory and its application, FD/FTC theory and its application, H1 control theory and its application, pulp and paper-making process computer control.

Jianxia Shou was born in Zhejiang, China, in 1964. She received the B.S. and M.S. degrees in computer science from Northeast University, China, in 1987 and 1990 respectively. From 1990 to 1993, she was with the Automation Instrument Company, Hangzhou, China, as a Embedded Software Developer. From 1994 to 2000, she was with the Hangzhou University of Electronic Technology, China, as a Lecturer. From 2001 to 2002, she was with the Sprint Corporation, Kansas, USA, as a Software Developer (Contractor). From 2003 to now, she was with the College of Software, Zhejiang University of Technology, China, as an Associate Professor. She has published more than 10 papers in some reputed journals and conferences. Her current research interests include learning control, artificial intelligence, etc.