# Projective Synchronization in A Circulant Partially Linear Chaotic System and Its Control

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## Abstract

Under different single variable driving coupled schemes, *Projective Synchronization ( $P_jS$ )* in coupled circulant partially linear chaotic systems is investigated in this paper. Both simple criteria for judging $P_jS$ occurrence and practical control strategy for the scale factor are discussed based on theoretical analysis. A typical chaotic system is also used to illustrate the proposed methods.
**Keyword**: Chaotic system, Circulant partially linear chaotic system, Projective synchronization.

## I. Introduction

During the last decades, synchronization in coupled chaotic system has become a topic of interest because of its potential applications. Different types of synchronization phenomena have been observed and experimentally verified in a variety of chaotic systems, see [2] and references therein. As a special type of generalized synchronization [10], projective synchronization, which means that the states of coupled subsystems could be synchronized up to a constant ratios, is firstly discussed in [3], and further investigated in [4]-[9].

$P_jS$ is characterized by a scale factor that defines a proportional relation between the synchronized systems. $P_jS$ results from the partial linearity of coupled chaotic system, and become the unique feature of partially linear systems [3]. The proportionality allows us to duplicate a chaotic system with different scales, while the topological characteristics (such as the Lyapunov exponents and fractal dimensions) of the two synchronized systems remain unchanged [4].

In this paper we investigate $P_jS$ in a circulant partially linear chaotic system newly proposed by Liu W-B and Chen G [1]. Conditions for the occurrence of $P_jS$ with different driving variable are proposed in section 1. Linear feedback control of the scale factor based on switch strategy in virtue of the system's inter-property is briefly discussed in section 2. The conclusion is given in section 3.

## II. $P_jS$ in the circulant partially linear chaotic system

Firstly, we define the circulant partially linear system as follows:

**Definition 1**

Assume the system is a set of ordinary differential equations, and can be described as:

$$\dot{W} = F(W, \rho) \tag{1}$$

where $W = (x_1, x_2, x_3)^T$, $\rho$ is system parameter. The state vector $W$ can be arbitrarily broken into two parts $(x_i, U)$. System (1) is called a circulant partially linear system. The system equation can be written in the following form:

$$\begin{aligned} \dot{U} &= M_i(x_i)U, \\ \dot{x}_i &= f(x_i; U). \end{aligned} \tag{2}$$

$M_i(x_i)$ is linearly dependent on $U$, and the equation for $x_i$ is nonlinearly related to the other variables, while the equation for the rate of change of vector $U$ is linearly related to $U$ through a Matrix $M$ that can depend on the variable $x_i$, as in (2).

Next, we will discuss the projective synchronization criteria [3] with different driving variable. For convenience, the discussion is illustrated using a newly proposed chaotic system in [1]. The newly proposed three-dimensional continuous autonomous chaotic system can display complex 2- and 4-scroll attractors in simulations. Its circuitry realization is also provided in [1]. The system dynamical equation is described by

$$\begin{cases} x_1 = ax_1 + d_1x_2x_3, \\ x_2 = bx_2 + d_2x_1x_3, \\ x_3 = cx_3 + d_3x_1x_2, \end{cases} \tag{3}$$

where $a, b, c, d_1, d_2, d_3$ are system parameters. The chaotic parameter regions that exhibit chaotic behaviors can be found in [1]. In the following discussion, the parameters are chosen as $a = 0.5, b = -10, c = -4, d_1 = 1, d_2 = -1, d - 3 = -1$, with which the system displays a 4-scroll attractor, as shown in Figure 1-4.

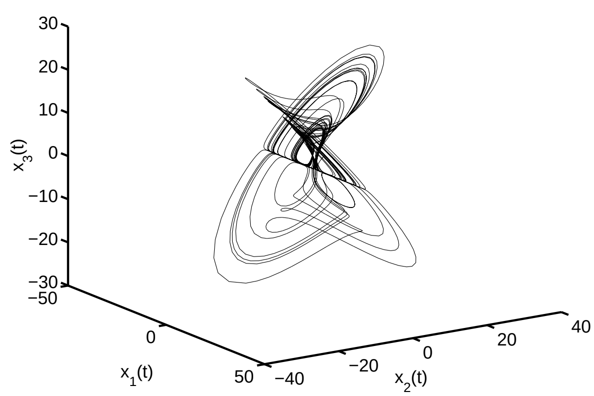


Figure 1. Attractor in 3D view.

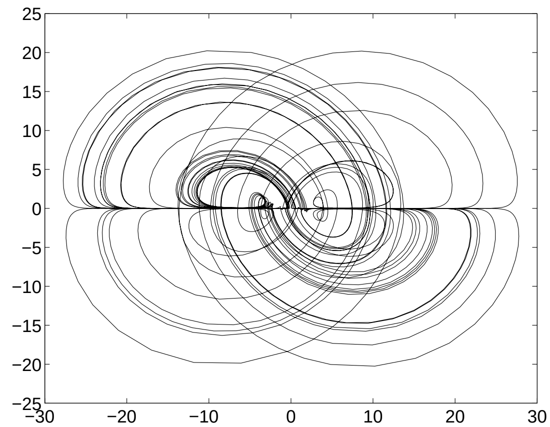


Figure 2. Phase portrait in 2D view on the $x_1 - x_2$ plane.

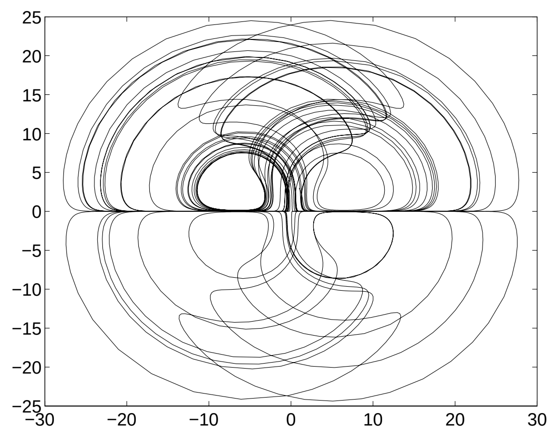


Figure 3. Phase portrait in 2D view on the $x_1 - x_3$ plane.

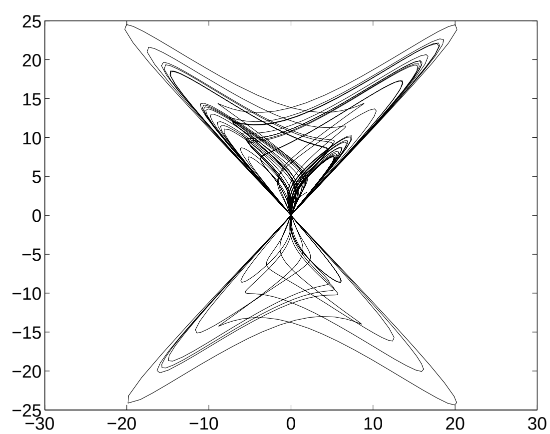


Figure 4. Phase portrait in 2D view on the $x_2 - x_3$ plane.

Notably, this new system is not equivalent in any sense with any other existing similar system, such as the classical Lorenz chaotic system, the Chen chaotic system, the Lü chaotic system,and so on, as mentioned in paper[1]. Obviously, it is a typical alternative partially linear system as defined before.

In what follows, we will investigate the projective synchronization for this system. Using $x_{3m}$ as the driving variable, the coupled system can be described as

$$\begin{cases} x_{1m} = ax_{1m} + d_1 x_{2m} x_{3m}, \\ x_{2m} = bx_{2m} + d_2 x_{1m} x_{3m}, \\ x_{3m} = cx_{3m} + d_3 x_{1m} x_{m2}, \\ x_{1s} = ax_{1s} + d_1 x_{2s} x_{3m}, \\ x_{2s} = bx_{2s} + d_2 x_{1s} x_{3m}, \\ x_{3s} = cx_{3m} + d_3 x_{1s} x_{2s}, \end{cases} \quad (4)$$

To derive the stability condition, one can investigate an error dynamics $e(t) = x_{1m}x_{2s} - x_{1s}x_{2m}$. Thus we can discuss in what parameter configuration in system(4) that the error dynamics converges to zero, namely, $\lim_{t\to\infty} e(t) = 0$.

In what follows, the Lyapunov function method is applied to derive the stability condition for projective synchronization.

Consider the following Lyapunov function

$$V = \frac{1}{2}e^2(t),$$

It satisfies

$$\begin{cases} V = 0, & if \quad e = 0; \\ V \neq 0, & if \quad e \neq 0. \end{cases}$$

Its time derivative is given by

$$\dot{V} = e(t)\dot{e}(t) = e^2(t)\{\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\}$$

If $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$, $\dot{V}$ satisfies

$$\begin{cases} \dot{V} = 0, & if \quad e = 0; \\ \dot{V} < 0, & if \quad e \neq 0. \end{cases}$$

That is, $V$ is positive definite and $\dot{V}$ is negative definite, i.e., $V$ will tend to zero as $t \to \infty$.

Similarly, if we choose $x_1$ as the driving variable and define the error as $e(t) = x_{2m}x_{3s} - x_{2s}x_{3m}$, the stability condition becomes $\frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} < 0$. If choose $x_2$ as the driving variable and define $e(t) = x_{1m}x_{3s} - x_{1s}x_{3m}$, the stability condition is $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_3}{\partial x_3} < 0$.

Therefore, we can derive the following theorem.

**Theorem 1**

*For the driving coupling circulant partially linear chaotic systems (1), projective synchronization will certainly occur via transfer only a single scale variable, if conditions* $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$ *;* $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_3}{\partial x_3} < 0$ *; and* $\frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} < 0$ *hold for all the parameters in possible parametric space.*

Theorem 1 can be easily used in practical applications, such as in secure communication, signal amplification in circuits, etc. We will compare situations with the change of scale factor and the synchronization speed under different driving variables numerically in the next section, in order to show the characteristic resulted from the circulant partially linearity. Fig.5-8 show different scale factors forming processes under different variable driving schemes with different initial conditions. In figure 9-11, the phase portraits of the master and slave systems are given for illustration with $x_3$ being the driving variable.

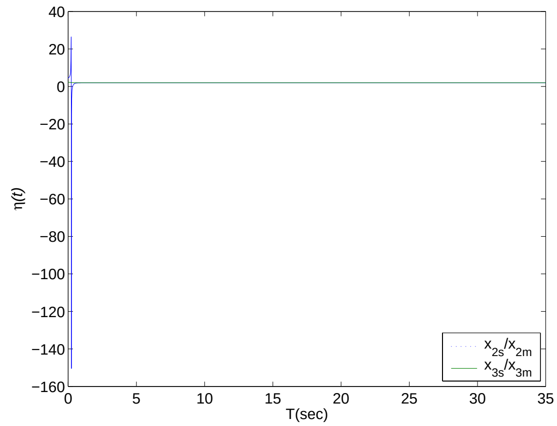


Figure 5. The change of scale factor under $x_1(t)$ driving, where initial condition is: $(1.0, 2.0, 3.0, 3.0, 9.0, 7.0)^T$, $\eta = 1.9716$.

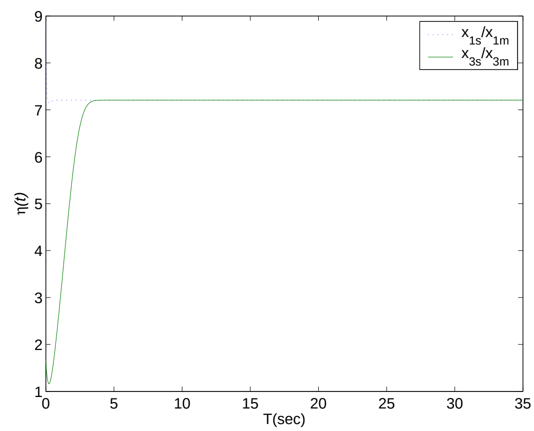


Figure 6. The change of scale factor under $x_2(t)$ driving, where initial condition is: $(1.0, 2.0, 3.0, 9.0, 3.0, 5.0)^T$, $\eta = 7.2061$.

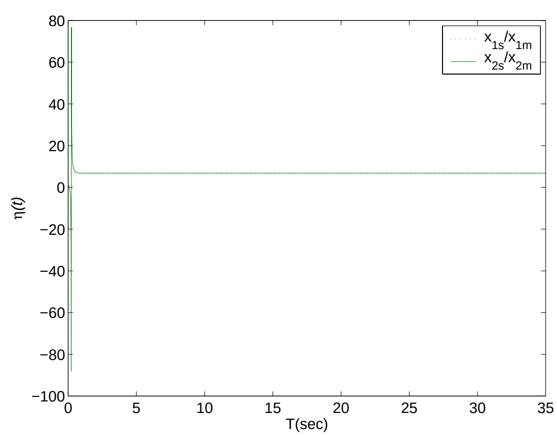


Figure 7. The change of scale factor under $x_3(t)$ driving, taking initial condition: $(1.0, 2.0, 3.0, 9.0, 3.0, 5.0)^T$, $\eta = 6.8175$.

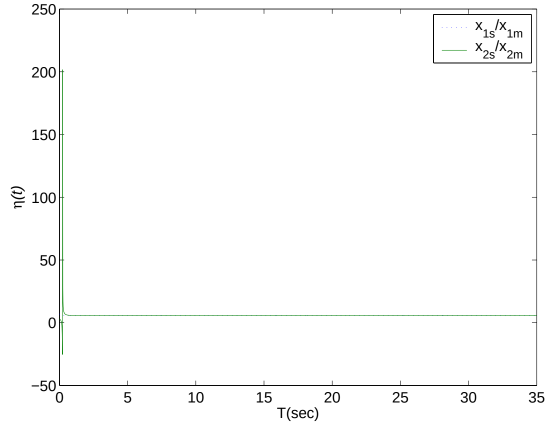


Figure 8. The change of scale factor under $x_3(t)$ driving, taking initial condition: $(1.0, 2.0, 3.0, 7.0, 6.0, 5.0)^T$, $\eta = 5.8361$, which is different from in Fig.7. It means the scale factor is unexpected and dependent on the initial conditions.

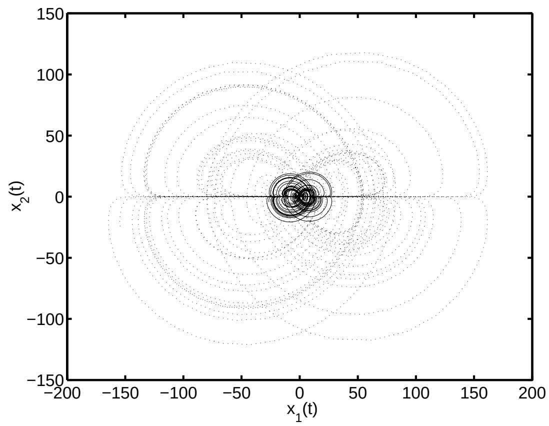


**Figure 9.** The 2-D phase graph under $x_3(t)$ driving, where initial condition is: $(1.0, 2.0, 3.0, 7.0, 6.0, 5.0)^T$, $\eta = 5.8361$, solid: master system, dotted: slave system.

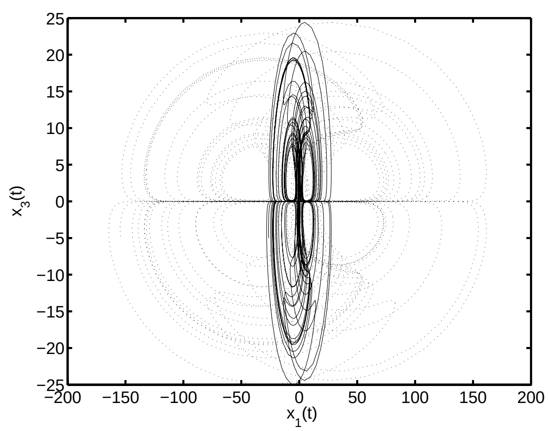


**Figure 10.** The 2-D phase graph under $x_3(t)$ driving, where initial condition is: $(1.0, 2.0, 3.0, 7.0, 6.0, 5.0)^T$, $\eta = 5.8361$, solid: master system, dotted: slave system.

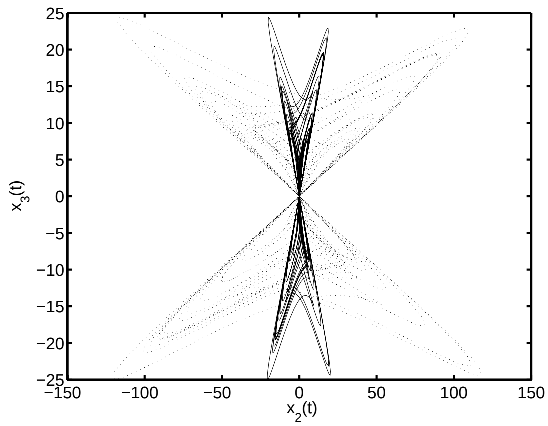


Figure 11. The 2-D phase graph under $x_3(t)$ driving, where initial condition is: $(1.0, 2.0, 3.0, 7.0, 6.0, 5.0)^T$, $\eta = 5.8361$, solid: master system, dotted: slave system.

## III. Controlling of the $P_jS$ scale factors

In this section, under different variable driving schemes, the practical control schemes of projective synchronization are investigated for alternating the formed scale factors. Recently, a simple but practical linear feedback control scheme is proposed in [5] and attracts much interest in related fields. Here the same scheme is used for controlling the scale factor of a circulant chaotic system. Here, we choose $x_{3m}$ as the driving variable to illustrate the practical control scheme.

Firstly, the equations of driving coupled systems are still described as (4). In order to manipulate the scaling factor, one should consider the good virtues of the typical circulant partially linear system, this has been proved to be very useful for the controlling task as used in [5]. So, to direct the slave system states evolving on the request of $P_jS$, one can take two steps to fulfill the task. We will use the situation transfer signal $x_{3m}$ for illustration in the following analysis.

Since under one-way driving by the coupling variable $x_{3m}$, one can observe the $P_jS$ occurrence and obtain the unexpected scale factor $\eta$; after the critical time, the phase synchronization persists all the time. After achieving first $P_jS$, one can introduce an additional controller (here named as *linear feedback plus driving coupling scheme*) to (4), switch on the controller after the critical time estimated in the first step, the scale factor will then be manipulated onto a new desired value. We will explain this in the following analysis part.

In order to know the precise behavior of this coupled system, we will investigate the time variation of the ratio $\frac{x_{1s}}{x_{1m}}$ (or $\frac{x_{2s}}{x_{2m}}$). For simplicity, introduce the cylindrical coordinates $(r, \theta, z)$, then $x_1 = rcos\theta, x_2 = rsin\theta, x_3 = x_3$ and $(\dot{r}, \dot{\theta}, x_3) = ((x_1x_1 + x_2x_2)/r, (x_1x_2 - x_2x_1)/r^2, x_3)$. In order to exclude the singularities of the ratio $\eta(t) = \frac{x_{1s}}{x_{1m}} = \frac{x_{2s}}{x_{2m}}$ in the following discussions, firstly, define the values of $\eta(t)$ in two special cases (it will be seen in the following discussion that the definitions have no conflicts with the whole analysis).

(1)Define $\eta = \infty$ when $r_m = 0, r_s \neq 0$ and $\eta = 0$ when $r_s = 0, r_m \neq 0$. In this case, only the master or the slave system converges to the origin.

(2)Define $\eta = \text{'}arbitrary\ \ value\text{'}$ when $r_m = 0$ and $r_s = 0$. In this case, both the master and slave system converge to the origin.

Secondly, from theorem 1, with the same condition, as $t \to \infty$,

$$\lim_{t\to\infty} e_p = \lim_{t\to\infty}(x_{1m}x_{2s} - x_{1s}x_{2m}) = 0$$

holds. We get

$$\lim_{t\to\infty} r_m r_s (\cos\theta_m \sin\theta_s - \cos\theta_s \sin\theta_m) = \lim_{t\to\infty} r_m r_s \sin(\theta_s - \theta_m) = 0,$$

Since $r_m r_s = 0$ does not always hold, it can be deduced that when $t \to \infty$, there must be $\sin(\theta_s - \theta_m) \to 0$. That means that the phase of the slave and the master will synchronize to a fixed slip of $\mathrm{K}\pi$, where $\mathrm{K} \in Z$, which also means the two system are phase locked by a phase difference of $0$ or $\mathrm{K}\pi$.

Thirdly, since

$$\frac{d}{dt}\left(\frac{r_s}{r_m}\right) = \frac{r_s}{r_m}\left(\frac{\dot r_s}{r_s} - \frac{\dot r_m}{r_m}\right) = \eta h(x_{3m}, \eta_m, \eta_s)\sin(\theta_s - \theta_m) \qquad (5)$$

where

$$h(x_{3m}, \theta_s, \theta_m) = \left[\left(\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_1}\right)\sin(\theta_s + \theta_m) + \left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right)\cos(\theta_s + \theta_m)\right]$$

Since $\lim_{t\to\infty}\sin(\theta_s - \theta_m) = 0$, one can deduce:

$$\lim_{t\to\infty}\frac{d}{dt}\left(\frac{r_s}{r_m}\right) = 0$$

It means $\lim_{t\to\infty}\frac{r_s}{r_m} = \overline{\eta}$, where $\overline{\eta} = Const$ (That obviously includes the singular cases defined before). Now it is declared that: $\lim_{t\to\infty}(r_s - \overline{\eta} r_m) = 0$.

From the equation (5) of $\eta(t) = \frac{r_s}{r_m}$, one can get

$$\eta(t) = \eta(0)\exp\left(\int_0^t h(x_{3m}, \theta_{s,}\theta_m)\sin(\theta_s - \theta_m)d\tau\right)$$

Where $\eta(0) = \frac{r_s(0)}{r_m(0)}$. That is to say, the scaling factor $\eta$ evolving with time and finally becoming variant, when $t > T_{Critical}$, where $T_{Cirtical}$ is the critical time at which $\sin(\theta_s - \theta_m)$ equals to zero. Since $\eta(t)$ changs with $x_{3m}, \theta_m, \theta_s$, it is unexpected in essence. But when $t > T_{Critical}$, $\eta(t)$ will remain unchangeable at $\overline{\eta} = \eta(0)\overline{C}$, where

$$\overline{C} = \exp\left(\int_0^{T_{Critical}} h(x_{3m}, \theta_{s,}\theta_m)\sin(\theta_s - \theta_m)d\tau\right),$$

since $\lim_{t\to\infty}\sin(\theta_s - \theta_m) = 0$ will always hold after the phase synchronization happens at the critical time (denoted aforementioned as $T_{Cirtical}$). In other words, the scaling factor is an unpredictable "constant", which is related to the initial conditions of the master and slave systems. Further study of this projective synchronization shows that the unpredictable scaling factor is controllable. That is to say, if proper controller is introduced, one can manipulate the factor to arbitrary designated value (that also means adjusting the amplitude of the slave system can be done).

As a particular case of those control strategies, the simple feedback control design (based on the switch strategy in virtue of the inter-property) can be derived as follows.

In order to manipulate the scaling factor, by introducing a controller to (5), the equation of $\eta(t)$ thus becomes

$$\frac{d}{dt}(\frac{r_s}{r_m}) = \dot{\eta} = \eta[h(x_{3m}, \eta_m, \eta_s)\sin(\theta_s - \theta_m) + \delta(t)] \tag{6}$$

where

$$\delta(t) = \begin{cases} 0, & if \quad t < T_{Critical} \\ \varepsilon(\eta^* - \eta). & if \ t \geq T_{Critical} \end{cases}$$

and the $\eta^*$ is the designated scaling factor, $\varepsilon$ is the feedback gain. Solving (6) when $t > T_{Critical}$. One can get

$$\eta(t) = \eta^*[1 + (\frac{\eta^*}{\overline{\eta}} - 1)\exp(-\varepsilon\eta^*(t - T_{Critical}))]^{-1}$$

Obviously, $\eta^*\varepsilon$ should always be satisfied to guarantee the success of controlling $\eta$ to $\eta^*$. Rewriting the controlled system in Descartes coordinates, we have the controlled system for $t > T_{Critical}$ as shown in (7), which can be seen as a typical 'linear feedback plus driving coupling systems' scheme as follows

$$\begin{cases} x_{1m} = ax_{1m} + d_1 x_{2m} x_{3m} + \varepsilon(x_{1s} - \eta^* x_{1m}), \\ x_{2m} = bx_{2m} + d_2 x_{1m} x_{3m} + \varepsilon(x_{2s} - \eta^* x_{2m}), \\ x_{3m} = cx_{3m} + d_3 x_{1m} x_{m2}, \\ x_{1s} = ax_{1s} + d_1 x_{2s} x_{3m}, \\ x_{2s} = bx_{2s} + d_2 x_{1s} x_{3m}, \\ x_{3s} = cx_{3m} + d_3 x_{1s} x_{2s}, \end{cases} \tag{7}$$

In practice, the whole control strategy now can be divided into two parts: firstly, design the uncontrolled system (4) and estimate the evolution time $T_{Cirtical}$ of achieving projective synchronization ; secondly, design the controlled system (7), setting the scaling factor $\eta^*$ as wanted; finally, switch on the controller after time $T_{Cirtical}$; if a new factor $\eta^{**}$ is needed, adjust the controller in (7) and repeating this procedure. Fig.12 shows an illustrative example for controlling $\eta_{unexpected}$ to $\eta_{desired}$.

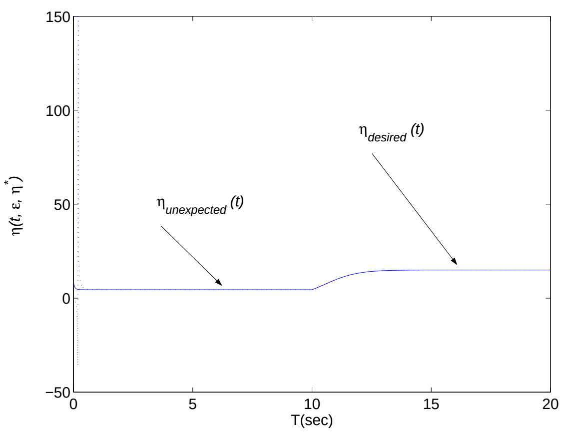


Figure 12. The graph of controlled scale factor under $x_3(t)$ driving, the controller switched on at 5th second, where initial condition is: $(1.0, 3.0, 5.0, 9.0, 1.0, 2.0)^T$, $\eta = 4.5036$, $\eta^* = 15.0$, solid line: $x_{1s}/x_{1m}$, dotted line: $x_{2s}/x_{2m}$.

## IV. Conclusion

Under different single variable driving coupled schemes, *Projective Synchronization* ( $P_jS$ ) in coupled circulant partially linear chaotic systems is investigated in this brief paper. Both simple criteria for judging $P_jS$ occurrence and practical switch control strategy for the scale factor are discussed based on theoretical analysis. A typical chaotic system, proposed by Liu W B, and Chen G. recently, is used to illustrate the proposed methods. Numerical experiments show the rightness and effectiveness of the theory analysis and control method. Further research and applications of such a typical category chaotic systems will be studied in near future.

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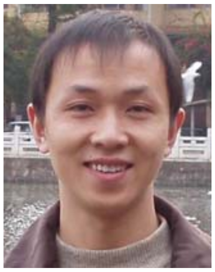

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